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AN APPLICATION OF THE INVARIANCE PRINCIPLE TO THE STUDENT HYPOTHESIS

BY

PAUL L. MEYER

TECHNICAL REPORT NO. 24

PREPARED UNDER CONTRACT N6onr-251 TASK ORDER 111
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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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By

Paul L. Meyer

1. Introduction and Summary.

The problem which is posed to the statistician in the formulation of the general decision problem as outlined in [1] reduces itself essentially to the choice of a pure or randomized decision procedure which will also be called a statistical strategy. The actual choice of a procedure depends on the criterion employed to decide how the risk is to be minimized.

Various criteria for selecting a decision procedure from a class of possible procedures have been investigated, with most attention having been given to the minimax and Bayes principle. Unfortunately, neither of these approaches is completely satisfactory; the former, since it assumes without must justification that nature -- the statistician's "opponent" -- will do its worst. The latter requires the knowledge of some a priori distribution, which is often not available. Hence in many cases, an 'optimal' solution to the general decision problem does not exist and we must accept somewhat less ambitious aims. This is analogous to the classical problem of testing hypotheses, where we often are unable to find uniformly most powerful tests and hence restrict ourselves to tests satisfying conditions such as unbiasedness, similarity, and invariance.

One way out of this dilemma is to construct a class of procedures which is optimal in the sense that no matter what criterion is used, one need not look outside it for selecting a procedure. Such a class is called essentially complete. It is often desirable to restrict oneself

further to a special class of statistical strategies from which to select an essentially complete class. This is done in order to reduce the number of possible procedures to be considered. Also this often leads to a simple characterization of an essentially complete class within this restricted class.

In this paper we construct an essentially complete class of invariant decision procedures for a type of problem arising from the observation of a normally distributed random variable.

More specifically, suppose x is normally distributed with mean μ and variance σ^2, both unknown. We consider the problem of making decisions concerning the quantity

$$p = \text{Prob}(x > 0) = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\infty} e^{-y^2/2} dy$$

where $\delta = \mu/\sigma$. We assume that the loss involved in making these decisions is a function only of p and hence only of δ, and does not depend on μ and σ individually. Decisions concerning the quantity p, whether they be in the form of its estimation or in the form of multi-decisions will be referred to as decisions of the Student hypothesis type.

A special case of the above problem occurs when we are testing, for example, the hypothesis H_o: $p = 1/2$ vs. H_1: $p > 1/2$. For if $p = 1/2$, δ and hence μ equals zero. Thus the above becomes a hypothesis involving the mean. This is the classical problem for which Student's t-test has been shown optimal.

By appealing to the invariance principle, it is shown here that the t-test is optimal for the more general situation of the Student hypothesis, given above. It is shown that these optimal procedures are monotone in t,

and form an essentially complete class of invariant decision procedures. Analogous results are obtained for the multi-variate case, in which Hotelling's T^2 replaces Student's t.

In order to demonstrate this optimality and monotonicity of the t-test for the general Student hypothesis, we appeal to a theorem proved by H. Rubin [7]. This theorem characterizes an essentially complete class of procedures as the class of monotone procedures. To prove this theorem, certain assumptions concerning the loss function and the action space are made. Also it is supposed that a real-valued random variable, depending on a real-valued parameter is observed; furthermore we assume that the distribution of this random variable has a monotone likelihood ratio.

We are able to reduce the general Student hypothesis type problem to the observation of a non-central t or non-central F variable, by restricting ourselves to invariant procedures. We show in this paper that both of these distributions possess monotone likelihood ratios, and hence the above theorem is applicable.

The mathematical formulation of the problem will be given on a much more general level than would be required to discuss the specific problems treated in this paper, and will follow quite closely a paper by Blackwell, Girshick, and Rubin [2] on the invariance principle. The reason for introducing the machinery given in [2] is to put the classical problems with the Student hypothesis into this new and more embracing framework.

2. The Student Hypothesis.

In many statistical problems, we are concerned with a normally distributed random variable, with mean μ and variance σ^2, both unknown, where the

consequences of the action taken based on a random sample depend only on the proportion of the area under this normal curve exceeding (or falling short of) a given number A. For example, in Industrial applications, the quality of a lot of goods may be measured by the fraction of the material that exceeds a given limit A, with respect to some normally distributed characteristic. On the basis of a sample of observations we may wish to either estimate the proportion or make decisions as to the disposition of the lot.

More specifically, we consider the problem of making decisions concerning

$$\text{Prob}(x > A) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_A^\infty e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad .$$

Taking $A = 0$, without loss of generality, and letting $y = x/\sigma$, we obtain

$$p = \text{Prob}(x > 0) = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^\infty e^{-y^2/2}\, dy \quad ,$$

where $\delta = \mu/\sigma$. We assume that the loss involved in making these decisions is a function only of p, and hence only of δ, and does not depend on μ and σ individually.

The classical problem for which Student's t is used, namely constructing significance tests for the mean of a normally distributed random variable, is clearly a special case of the above. For if $p = 1/2$, δ and hence μ equals zero, and the testing of H_0: $p = 1/2$ vs. H_1: $p \neq \frac{1}{2}$ becomes a test involving the mean. In order to prove optimality of the t-test, even in this special case, we assume that the consequence of a wrong decision depends on δ and not on μ and σ individually.

Various optimal properties for the test based on Student's t have been demonstrated for this special case. For instance, we know that if we test

(i) H_o: $\mu = 0$

H_1: $\mu > 0$,

the u.m.p. test is

$$\varphi(t) = 1 \quad \text{if} \quad t < c$$

$$\varphi(t) = 0 \quad \text{if} \quad t > c \quad ,$$

where $\varphi(t)$ = Prob. accepting H_o, given t, or

(ii) H_o: $\mu = 0$

H_1: $\mu \neq 0$,

for which no u.m.p. test exists, but the u.m.p. unbiased test is

$$\varphi(t) = 1 \quad \text{if} \quad |t| < k$$

$$\varphi(t) = 0 \quad \text{if} \quad |t| > k \quad .$$

When we restrict ourselves to invariant procedures for testing the above hypothesis, it has been shown that the t-test yields the best invariant procedure. Recently the admissibility of this best invariant procedure has been shown by Lehmann and Stein [4]. The power function, which was used as a criterion for loss is a function of δ only and hence is of the type considered.

In this paper we show, that by appealing to the invariance principle, optimal properties of the t-test hold in the general problem of the Student hypothesis, as given above. We may, for instance, partition the δ-axis into k intervals (non-overlapping) $I_1, \ldots, I_k$ and test the hypotheses H_i: $\delta \in I_i$, $i = 1, \ldots, k$, i.e., consider a multi-decision problem. If we then assume that for a fixed decision, the loss function depends on δ and possesses certain monotonicity properties, we can by restricting ourselves to invariant procedures, find a strategy based on t which is uniformly as good as or better

than any other given invariant procedure. Furthermore, a constructive method for finding such a better procedure is given in [1]. These procedures based on t, will turn out to be monotone. When only 2 actions are involved, the essentially complete class of invariant procedures (i.e., the monotone procedures based on t) is minimal. We may also wish to estimate the quantity p. Here again, the invariant estimate which is optimum will be a monotone function of t.

In considering invariant procedures, many others besides t suggest themselves. For example, the statistic $H = \bar{x}/R$, where R is the sample range is commonly used in quality control applications. It was shown in [8], that for $n \leq 10$, H yields an excellent approximation to t in terms of the power function. Naturally, if we take computational and time costs into consideration it may well be that for small samples the use of H is better than that of t. Ignoring these costs, the above discussion shows that procedures based on t dominate those based on H.

3. Mathematical Formulation and Definitions.

In this section we shall discuss the mathematical framework within which our results will be stated. Also we shall define rigorously various concepts which have been referred to rather loosely in the preceding sections.

Although most of the concepts are discussed in great detail elsewhere, as, for example, in [1] and [2], some of the basic ideas will be recapitulated here for the sake of completeness.

We assume that we are given the following:

(1) A sample of space $\mathcal{Z} = (Z, \mathcal{B}, \Omega, P)$ where Z is the space of outcomes of an experiment, $\mathcal{B}$ a Borel field of subsets of Z, Ω an arbitrary parameter space, and P a function defined on $\mathcal{B} \times \Omega$, so that for

each $\omega \epsilon \Omega$, P_ω is a probability measure on $\mathcal{B}$. We shall write, for $S \in \mathcal{B}$, $\omega \epsilon \Omega$, $P_\omega(S) = P(S|\omega)$.

(2) An action space A and Borel field $\mathcal{a}$ of subsets of A.

(3) A loss function L defined on $\Omega \times A$ which is $\mathcal{a}$-measurable for each $\omega \epsilon \Omega$. For any $\omega \epsilon \Omega$, $a \in A$, $L(\omega, a)$ represents the loss to the statistician if nature is in state ω and he chooses action a. We may assume $L(\omega, a) \geq 0$.

(4) A class of randomized decision functions $\mathcal{N}$ so that for each $\nu \epsilon \mathcal{N}$, $z \in Z$, ν_z is a probability measure on $\mathcal{a}$. In particular, for $E \in \mathcal{a}$, $z \in Z$, $\nu_z(E)$ is the probability of taking action E, on observing z.

In most of our applications we shall only need to consider the class D of pure decision procedures, where each d maps Z into A.

(5) A statistical game $G = (\Omega, \mathcal{N}, \rho)$ where ρ is the risk function defined by

$$\rho(\omega, \nu) = \int_Z \int_A L(\omega, a) d\nu(a|z) dP(z|\omega) \quad .$$

In terms of these concepts, we now define:

A class C of decision procedures is essentially complete if for any procedure ν', there exists a procedure $\nu \in C$ so that $\rho(\omega, \nu) \leq \rho(\omega, \nu')$ for all $\omega \epsilon \Omega$. If for any $\nu' \notin C$ we can find $\nu \in C$ so that $\rho(\omega, \nu) \leq \rho(\omega, \nu')$ for all ω, with strict inequality for at least one ω, then we say C is complete.

Clearly the construction of a complete class C is extremely desirable, since we need not look outside it for selecint procedures for a particular statistical game.

Next we introduce the invariance principle as we shall use it in our applications. This principle has been used to advantage in various statistical problems, as is mentioned in a paper by E. Lehmann [5]. The formulation given here following the previously mentioned paper [2], is different from the earlier ones in that it considers invariance from the more general point of view of decision theory.

Let $\mathcal{G}$ be a group. With each $g \in \mathcal{G}$, are associated functions g_Z, g_Ω, and g_A, defined on Z, Ω, and A respectively, so that $g \to g_Z$ is a homomorphism of $\mathcal{G}$ into the transformation group on Z, with similar interpretations for the correspondence $g \to g_\Omega$ and $g \to g_A$; i.e., with each element of the group is associated in a 1-1 way an element which maps Z onto itself, A onto itself, and Ω onto itself.

We shall only deal with a particular type of group, namely admissible, which we now define.

<u>Definition</u>. The group $\mathcal{G}$ and its associated functions g_Z, g_Ω, and g_A are admissible with respect to the game $G = (\Omega, D, \rho)$ if:

(i) for each $g \in \mathcal{G}$, g_Z and g_A are measurable with respect to $\mathcal{B}$ and $\mathcal{Q}$ respectively;

(ii) for each $g \in \mathcal{G}$, $S \in \mathcal{B}$, $\omega \in \Omega$,

$$P(g_Z(S) \mid g_\Omega(\omega)) = P(S \mid \omega);$$

(iii) for each $g \in \mathcal{G}$, $a \in A$, $\omega \in \Omega$,

$$L(g_\Omega(\omega), g_A(a)) = L(\omega, a).$$

We assume in what follows, that only admissible groups are considered.

The purpose of introducing invariance at all was in order to reduce the number of possible procedures which might have to be considered, and restrict ourselves only to invariant procedures, which we now define.

Definition. A pure decision function $d \in D$ is invariant under $\mathcal{G}$ if for all $g \in \mathcal{G}$, z Z

$$d(g_Z(z)) = g_A(d(z)) \quad .$$

A randomized decision function $\nu \in \mathcal{N}$ is said to be invariant under $\mathcal{G}$, if for every $z \in Z$, $E \in \mathcal{A}$, and $g \in \mathcal{G}$,

$$\nu(g_A(E) | g_Z(z)) = \nu(E/z) \quad .$$

These definitions simply say that a decision procedure is invariant if the same action results from observing z after it has been operated on by $\mathcal{G}$, e.g., observing $g_Z(z)$.

We shall next define another concept that arises in studying invariant procedures, namely that of an orbit. This is very closely related to the more familiar concept of a maximal invariant, as we shall point out below.

Definition. Let $z_o \in Z$ be a fixed element. Then we call $K_{z_o} = \{z: z = g_Z(z_o) \text{ for some } g \in \mathcal{G}\}$ the orbit generated by z_o.

As we vary z_o over Z, we obtain the class $\mathcal{K}$ of orbits on Z; this is clearly a partition of Z, e.g., if we define $z_1 \sim z_2$ for z_1, z_2 in the same orbit, then $\sim$ defines an equivalence relation over Z. To tie this in with the concept of a maximal invariant, we use the following definition.

Definition. A function f defined on Z is a maximal invariant if

(i) $f(g_Z(z)) = f(z)$ for all $g \in \mathcal{G}$, $z \in Z$;

(ii) $f(z_2) = f(z_1) \Rightarrow$ there exists a $g \in \mathcal{G}$ so that $z_2 = g_Z(z_1)$.

It is clear that a maximal invariant is constant on each orbit and assumes different constant values on different orbits. There may be many functions f which are maximal invariants for a group $\mathcal{G}$; each, however, induces the same partition on Z, namely the class of orbits. Hence we may identify orbits and maximal invariants, as we shall do later.

Needless to say, the same concepts discussed above for Z, apply also to Ω.

We shall now summarize a few of the main results as set forth in [2]. Before getting involved in a maze of notation, we shall briefly outline what we are attempting to do.

We start out with a statistical game $G = (\Omega, \mathcal{N}, \rho)$, where both the spaces Z and Ω may be of rather complicated form. We introduce an equivalent game G^* (for the definition of equivalence of games, see [1]), in which nature chooses an orbit of Ω and an element $g \in \mathcal{G}$, and the statistician observes an orbit K. If we restrict ourselves to invariant procedures (with respect to an admissible group), it turns out that the risk ρ^* in the equivalent game G^* does not depend on the choice of $g \in \mathcal{G}$.

We now give a brief resumé of the mathematics involved in the above reduction. We assume given an admissible group $\mathcal{G}$ and a statistical game $G = (\Omega, \mathcal{N}, \rho)$:

(i) Consider the class of orbits on Ω, say ϑ, i.e., $\Theta \in \vartheta$ implies

$$\Theta = \{\omega: \text{ for some } g,\ \omega = g_\Omega(\omega_0)\} \text{ for fixed } \omega_0.$$

Now fix any $\Theta \in \vartheta$ and let Ψ be a selection function defined on ϑ, and taking values in Θ, i.e., for each Θ, Ψ chooses a point $\Psi\Theta \in \Theta$. Define

$$P^*(S|g,\Theta) = P(S|g_\Omega(\Psi\Theta))$$

$$L^*(g,\Theta,a) = L(g_\Omega(\Psi\Theta),a) \quad ,$$

then

$$G = (\Omega, \mathcal{N}, \rho) \sim G^* = (\Omega^*, \mathcal{N}, \rho^*)$$

where

$$\Omega^* = \mathcal{G} \times \mathcal{N} \quad ,$$

and

$$\rho^*(g,\Theta,\nu) = \int_Z\int_A L^*(g,\Theta,a)\,d\nu(a|z)\,dP^*(z|g,\Theta) \quad .$$

(ii) Assume

$$\mathcal{J} = \{g: g_Z(z_o) = z_o\} = \{(e)\}$$

where e is the identity element of $\mathcal{G}$. Then there exists a 1-1 function $\gamma : \mathcal{G} \rightarrow K$, defined by $\gamma(g) = g_Z(z_o)$, for z_o fixed $\in K$. Hence there is induced a probability distribution Γ_K over $\mathcal{G}$, since a distribution is assumed to exist over K. Also, if ν is an invariant procedure, ρ^* is independent of the choice of g and we may as well use e. Thus we can write

$$\rho^*(e,\Theta,\eta) = \int_{\mathcal{K}}\int_A\int_{\mathcal{G}} L^*(e,\Theta,g_A(a))\,d\Gamma_K(g|\Theta)\,d\eta(a|K)\,dQ(K|\Theta)$$

where $d\eta_K$ is a probability distribution over A for given K, and dQ is a probability distribution over K for given Θ.

(iii) In the special case in which $g_A(a) = a$ the expression for ρ^* in (ii) simplifies to

$$\rho^*(e,\Theta,\eta) = \int_{\Theta}\int_A L^*(e,\Theta,a)\,d\eta(a|K)\,dQ(K|\Theta) \quad .$$

This reduction is possible since the integration over $\mathcal{G}$ was eliminated as

$$\int_{\mathcal{G}} d\Gamma_K(g) = 1 \quad .$$

It is this form which will occur in most of our applications.

(iv) Suppose that f and λ are maximal invariants on Z and Ω respectively. Then we may write (making appropriate notational changes since originally the functions involved were defined for different arguments):

$$\overline{\rho^*}(e,\lambda,\overline{\eta}) = \int\int_A \overline{L^*}(e,\lambda,a)\,d\overline{\eta}(a|f)\,d\overline{Q}(f|\lambda) \quad .$$

4. Characterization of an Essentially Complete Class.

We shall now state the main theorem used in obtaining essentially complete classes of decision procedures. It is a theorem proved by H. Rubin [7].

Before stating the main result, we introduce a few more concepts.

Definition. Let Z be a real-valued random variable and let A, the action space, be a closed subset of the real line. Then a monotone procedure d: $Z \rightarrow A$ is defined by:

$$\left.\begin{array}{ll} x,y \in Z \quad , & x > y \\ d(x) = a_1 \ , & d(y) = a_2 \end{array}\right\} \Longrightarrow a_1 \geq a_2 \quad .$$

If A is finite, e.g., $A = (a_1, \ldots, a_k)$, a monotone procedure is characterized by a set of numbers $x_o \leq x_1 \leq \ldots \leq x_k$ so that action i is taken if and only if the outcome is a point in $[x_{i-1}, x_i)$. (See, for example, [1], Chapter 7.)

Definition. Let Z be a real-valued random variable. Suppose the probability distribution of Z, p_ω depends on a real-valued parameter ω. Then $p_\omega(z)$ is said to have a monotone likelihood ratio if for $z_1 \geq z_2$, $\omega_1 \geq \omega_2$, we have:

$$p(z_1 | \omega_1) p(z_2 | \omega_2) \geq p(z_1 | \omega_2) p(z_2 | \omega_1) \quad .$$

As we have mentioned in some of the introductory sections of this paper, we shall restrict ourselves to a special class of loss functions, which we now describe.

Let A be a closed subset of the real line, and assume the parameter space to be an interval, say (a,b).

Suppose $\inf_{a \in A} L(\omega, a)$ is assumed; the point at which it is assumed clearly depends on ω and we denote it by $q(\omega)$. It is obvious that $q(\omega)$ need not be unique, since there may be a whole set of values of ω which

yields inf $L(\omega, a)$. We simply let $q(\omega)$ be a point at which this inf is assumed.

We further suppose

(i) $q(\omega)$ is increasing in ω,

(ii) $$\left.\begin{matrix} a \le a' \le q(\omega) \\ q(\omega) \le a' \le a \end{matrix}\right\} \Rightarrow L(\omega, a\ ) \le L(\omega, a) \quad \text{for all } \omega.$$

If A is finite, say $A = (a_1, \ldots, a_k)$, this reduces to being able to label $a_1, \ldots, a_k$ so that $\Omega = (a,b)$ can be subdivided into k consecutive subintervals $I_1, \ldots, I_k$ (some of which may be empty) with $\bigcup_{j=1}^{k} I_i = \Omega$, so that

(i) for all $\omega \in I_i$, $L(\omega, i) = \min_j L(\omega, j)$

(ii) $$\left.\begin{matrix} S \le j \le i \\ i \le j \le S \end{matrix}\right\} \Rightarrow L(\omega, j) \le L(\omega, S) \quad \text{for all } \omega \in I_i .$$

We can now state the main theorem of this section.

Theorem. Let Z be a real-valued random variable, Ω a subset of the real line, and A a closed subset of the real line. Suppose that p_ω, the distribution of Z, depends on ω in such a way that it has a monotone likelihood ratio. Suppose further that the loss $L(\omega, a)$ satisfies the conditions stipulated above. Then an essentially complete class of decision procedures can be characterized as the class of monotone procedures.

This theorem is proved in [1] for the class of exponential distributions and finite A. The extension to the above form is contained in an unpublished paper by Rubin.

In order to apply this theorem to the construction of essentially complete classes, we must reduce the problem to the observation of a real-valued random variable having a distribution with a monotone likelihood

ratio (supposing the other assumptions are fulfilled). It is for this reduction that we make use of invariance, e.g., restricting ourselves to invariant procedures, and show that in certain cases we can reduce the problem to the form for which the above theorem holds.

To do this for the specific case where we deal with observations obtained from a univariate or multi-variate normal populations we need two preliminary results, which will be developed in the next section.

5. Monotonicity of the Non-central t and Non-central F Distribution.

We first consider the case of the non-central t-distribution. Suppose z and w are independently distributed random variables, z being $N(\delta, 1)$ and w being χ^2_k. Then the joint distribution of z and w is

$$f(z,w) = \frac{1}{\sqrt{2\pi}\, 2^{k/2}\, \Gamma(\frac{k}{2})}\, e^{-\frac{1}{2}(z-\delta)^2}\, w^{\frac{k-2}{2}}\, e^{-w/2} \quad .$$

Letting $t = z/\sqrt{w/k}$, we obtain for the distribution of t:

$$p(t\,|\,\delta) = c\int_0^\infty e^{-1/2(\sqrt{\frac{w}{k}}\,t-\delta)^2}\, w^{\frac{k-1}{2}}\, e^{-w/2}\, dw \quad .$$

We now shall prove:

Theorem. The non-central t distribution $p(t\,|\,\delta)$ has a monotone likelihood ratio.

Proof. The proof of this theorem has been given in [3] and [4]; the proof given here is a slight modification of the one found in [4].

Let

$$F(t) = \frac{p(t\,|\,\delta_1)}{p(t\,|\,\delta_2)} \quad ,$$

where $\delta_1 > \delta_2$. We must show that $t_1 > t_2 \Rightarrow F(t_1) > F(t_2)$. Since F is continuous in t, only 2 cases arise:

(i) $\quad t_2 < t_1 < 0$

(ii) $\quad t_1 > t_2 > 0$.

We shall prove (i); e.g., suppose $t < 0$, Let $-t\sqrt{w} = v$. Then one obtains

$$F(t) = \frac{\int_0^\infty e^{-\frac{1}{2}\left(\frac{v}{\sqrt{k}} + \delta_2\right)^2} \left(\frac{-v}{t}\right)^{k-1} e^{-\frac{v^2}{2t^2}} \frac{2v}{t^2}\, dv}{\int_0^\infty e^{-\frac{1}{2}\left(\frac{v}{\sqrt{k}} + \delta_2\right)^2} \left(\frac{-v}{t}\right)^{k-1} e^{-\frac{v^2}{2t^2}} \frac{2v}{t^2}\, dv}$$

$$= \frac{\int_0^\infty e^{-\frac{v\delta_1}{\sqrt{k}}} e^{-\frac{v^2}{2t^2}} f(v) dv}{\int_0^\infty e^{-\frac{v\delta_2}{\sqrt{k}}} e^{-\frac{v^2}{2t^2}} f(v) dv}$$

where

$$c = e^{-\frac{1}{2}(\delta_1^2 - \delta_2^2)} > 0$$

and

$$f(v) = v^k e^{-\frac{v^2}{2k}} .$$

Hence

$$F(t) = F^*(\gamma) = \frac{c\int_0^\infty f(v) e^{-\frac{v\delta_1}{\sqrt{k}}} e^{-\gamma v^2} dv}{\int_0^\infty F(v) e^{-\frac{v\delta_2}{\sqrt{k}}} e^{-\gamma v^2} dv}$$

where

$$\gamma = \frac{1}{2t^2} .$$

Now as $t < 0$ increases, t^2 decreases, and so γ increases. Hence we must show that F^* is increasing in γ. Let $0 < \gamma_1 < \gamma_2$ and define

$$A = \frac{1}{c}\left[F^*(\gamma_2) - F^*(\gamma_1)\right] .$$

Therefore

$$A = \frac{\int_0^\infty f(v) e^{-\frac{v\delta_1}{\sqrt{k}} - \gamma_2 v^2} dv}{\int_0^\infty f(v) e^{-\frac{v\delta_2}{\sqrt{k}} - \gamma_2 v^2} dv} - \frac{\int_0^\infty f(v) e^{-\frac{v\delta_1}{\sqrt{k}} - \gamma_1 v^2} dv}{\int_0^\infty f(v) e^{-\frac{v\delta_2}{\sqrt{k}} - \gamma_1 v^2} dv}$$

$$= \int_0^\infty e^{-\frac{v}{\sqrt{k}}(\delta_1 - \delta_2)} \left[g_2(v) - g_1(v)\right] dv$$

where

$$g_1(v) = \frac{f(v) e^{-\gamma_1 v^2 - \frac{\delta_2}{\sqrt{k}} v}}{\int_0^\infty f(z) e^{-\frac{z\delta_2}{\sqrt{k}} - \gamma_1 z^2} dz} .$$

However,

$$\int_0^\infty \left[g_2(v) - g_1(v)\right] dv = \int_0^\infty \left[\frac{f(v) e^{-\gamma_2 v^2 - \frac{v\delta_2}{\sqrt{k}}}}{\int_0^\infty f(z) e^{-\frac{z\delta_2}{\sqrt{k}} - \gamma_2 z^2} dz} - \frac{f(v) e^{-\gamma_1 v^2 - \frac{\delta_2}{\sqrt{k}} v}}{\int_0^\infty f(z) e^{-\frac{z\delta_2}{\sqrt{k}} - \gamma_1 z^2} dz}\right] dv$$

$$= 0 .$$

Also,

$$\frac{g_2(v)}{g_1(v)} = c' e^{-(\gamma_2 - \gamma_1) v^2} , \qquad c' > 0 .$$

Therefore, v increasing implies g_2/g_1 decreasing. Hence by continuity argument, we can find an M so that

$$\frac{g_2(v)}{g_1(v)} \begin{array}{ll} > 1 & \text{if } 0 \leq v \leq M \\ < 1 & \text{if } M < v < \infty \end{array} .$$

Thus applying the mean value theorem, we may write:

$$A = \int_0^M e^{-\frac{v}{\sqrt{k}}(\delta_1-\delta_2)}[g_2(v)-g_1(v)]dv + \int_M^\infty e^{-\frac{v}{\sqrt{k}}(\delta_1-\delta_2)}[g_2(v)-g_1(v)]dv$$

$$= e^{-\frac{v_0}{\sqrt{k}}(\delta_1-\delta_2)}\int_0^M [g_2(v)-g_1(v)]dv + e^{-\frac{v_1}{\sqrt{k}}(\delta_1-\delta_2)}\int_M^\infty [g_2(v)-g_1(v)]dv$$

where

$$0 < v_0 < M$$

$$M < v_1 < \infty \quad .$$

Hence

$$A > e^{-\frac{v_1}{\sqrt{k}}(\delta_1-\delta_2)}\int_0^M [g_2(v)-g_1(v)]dv + e^{-\frac{v_1}{\sqrt{k}}(\delta_1-\delta_2)}\int_M^\infty [g_2(v)-g_1(v)]dv$$

$$= e^{-\frac{v_1}{\sqrt{k}}(\delta_1-\delta_2)}\int_0^\infty [g_2(v)-g_1(v)]dv = 0 \quad .$$

This proves our contention.

The proof for (ii): $t_1 > t_2 > 0$ is, except for a few minor changes, the same.

We shall now derive a similar property for the non-central F distribution.

<u>Theorem</u>. Let $G = u/v$ where u and v are independently distributed according to the non-central χ^2 distribution with r d.f. and χ^2 distributions with s d.f. respectively. Then the probability distribution of G, $p(G|\lambda)$ has a monotone likelihood ratio, where λ is the non-centrality parameter. The distribution $p(G|\lambda)$ is called the non-central F distribution.

Proof. The expression for $p(G|\lambda)$ is given by

$$p(G|\lambda) = \sum_{m=0}^{\infty} e^{-\lambda}\left(\frac{G}{1+G}\right)^{\frac{r+2m-2}{2}} \left(\frac{1}{1+G}\right)^{\frac{s+2}{2}} \frac{\lambda^m}{m!} k(r,s,m)$$

where

$$k = [B(\tfrac{r}{2} + m_1, \tfrac{s}{2})]^{-1} \quad .$$

Letting $u = \frac{G}{1+G}$, which is clearly an increasing function of G, we have:

$$p(G|\lambda) = p^*(u|\lambda) = (1-u)^{\frac{s+2}{2}} u^{\frac{r-2}{2}} \sum_{m=0}^{\infty} e^{-\lambda} u^m \frac{\lambda^m}{m!} k(m) \quad .$$

For $\lambda_1 > \lambda_2$, let

$$F(u) = \frac{p^*(u|\lambda_1)}{p^*(u|\lambda_2)} \quad .$$

Hence

$$F(u) = e^{-(\lambda_1-\lambda_2)} \frac{\sum_{m=0}^{\infty} u^m \frac{\lambda_1^m}{m!} k(m)}{\sum_{m=0}^{\infty} u^m \frac{\lambda_2^m}{m!} k(m)} = \frac{cF_1(u)}{F_2(u)} \qquad \text{say.}$$

Differentiating F with respect to u yields:

$$F'(u) = \frac{c[F_2(u)F_1'(u)-F_1(u)F_2'(u)]}{\{F_2(u)\}^2} \quad .$$

The functions F_i may be differentiated termwise and one obtains:

$$F_i'(u) = \sum_{n=0}^{\infty} n \frac{\lambda_i^n}{n!} u^{n-1} k(n) \quad .$$

Thus

$$F_2(u)F_1'(u)-F_1(u)F_2'(u) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^{m+n-1} n a_m a_n [\lambda_2^m \lambda_1^n - \lambda_1^m \lambda_2^n]$$

where

$$a_m = \frac{k(m)}{m!}$$

We may write the product of the above series in terms of the double series indicated, since we are dealing with uniformly convergent series of positive terms. Now the above double series may be written as (omitting the arguments):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \qquad ,$$

since for $m = n$, the argument is zero.

More generally, we may write (if interchange of summation is permissible, as it is in our case):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m,n) = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} f(m,n)$$

$$= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} f(t+n,n)$$

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} f(m,n) = \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} f(m,t+m) \qquad .$$

Using these results in the above expression, we have:

$$F_2(u)F_1'(u)-F_1(u)F_2'(u)$$

$$= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} n\,a_{t+n}\,a_n u^{t+2n-1}[\lambda_2^{t+n}\lambda_1^{n} - \lambda_1^{t+n}\lambda_2^{n}]$$

$$+ \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} (t+m)a_m a_{m+t} u^{t+2m-1}[\lambda_2^{m}\lambda_1^{t+m} - \lambda_1^{m}\lambda_2^{t+m}]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n a_{n+m} u^{m+2n-1}\left\{n[\lambda_2^{m+n}\lambda_1^{n} - \lambda_1^{m+n}\lambda_2^{n}] + (n+m)[\lambda_2^{n}\lambda_1^{m+n} - \lambda_1^{n}\lambda_2^{m+n}]\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n a_{m+n} u^{m+2n-1} m(\lambda_1\lambda_2)^n[\lambda_1^{m} - \lambda_2^{m}]$$

$$> 0$$

which proves the assertion.

6. Applications to Normally Distributed Variables.

We shall now apply some of the principles of the previous sections to problems arising in inference based on observations from normally distributed random variables. In particular, we shall study the t-test in the light of the discussion carried on earlier.

Let $x_1,\ldots,x_N$ be N independent observations from a random variable with distribution $N(\mu,\sigma^2)$. Since the statistics $\bar{x}=\frac{1}{N}\sum_1^N x_i$ and $s^2=\sum_1^N (x_i-\bar{x})^2$ are jointly sufficient for μ, σ^2, we may consider Z, the space of outcomes to consist of the pairs $\{(\bar{x},s),\ s>0\}$ while the parameter space Ω is representable as $\{(\mu,\sigma),\ \sigma>0\}$. Defining $t=\bar{x}/s$ and $\delta=\mu/\sigma$, we note that we may equivalently write for $z\in Z$, $z=(ts,s)$, and for $\omega\in\Omega$, $\omega=(\delta\sigma,\sigma)$.

We note that the sufficiency principle has made it possible to reduce our observation $(x_1,\ldots,x_N)$ from a point in N-dimensional space to one in 2-dimensional space. Using the invariance principle, we shall further reduce this to the observation of a real-valued random variable, in order to apply the above theory on complete classes.

We have for the distribution of $z=(\bar{x},s)$:

$$p_\omega(\bar{x},s)=\frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{N}{2}(\bar{x}-\mu)^2}\,\frac{2}{2^{\frac{N-1}{2}}\sigma^N\,\Gamma(\frac{N-1}{2})}\,s^{N-2}\,e^{-\frac{s^2}{2\sigma^2}}\quad,\ \text{for } s>0.$$

Now $t'=\sqrt{N(N-1)}\,\frac{\bar{x}}{s}$ has the non-central t distribution, say $p(t'|\delta)$, with non-centrality parameter $\sqrt{N}\,\delta$.

We first consider the group $\mathcal{G}_1$

$$g_Z(\bar{x},s) = g\bar{x}\ ,\ gs$$

$$g_\Omega(\mu,\sigma) = g\mu\ ,\ g\sigma \qquad\qquad g > 0$$

$$g_A(a) = a \quad .$$

This type of group operates, for example, when data is subjected to change of scale. When dealing with $\mathcal{G}_1$, we assume the loss L to be of the form

$$L(\omega,a) = L((\delta,1)a) \quad ,$$

i.e., L depends on ω only through δ. As we have stated before, this occurs, for instance, in the classical case of testing hypotheses, where the loss is measured in terms of the power of the test.

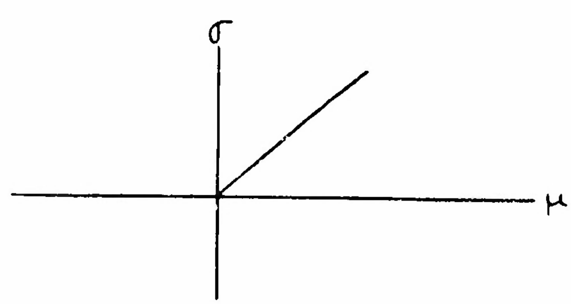
We now consider various problems which remain invariant with respect to $\mathcal{G}_1$.

It is easily checked that $\mathcal{G}_1$ is an admissible group for $G = (\Omega, \mathcal{N}, \rho)$ where

$$\rho = \int\int L(\omega,a)d\nu(a|z)dP(z|\omega) \quad .$$

The orbits for this group are easily determined.

Fixing $\omega_o = (\delta,1)\in\Omega$ we obtain for the orbit $\Theta_{\omega_o} = \{\omega:\ \delta = c\}$, where c is a constant. Geometrically, this is a ray through the origin in the μ- σ plane:

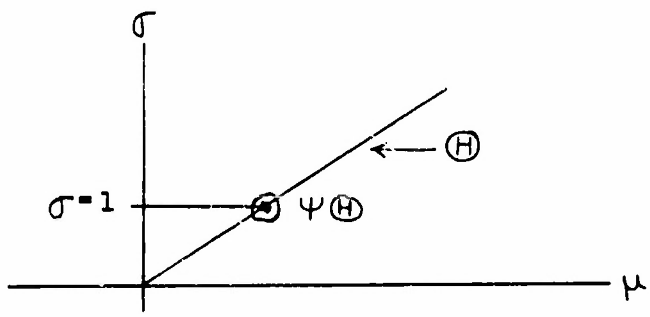


Thus the class of orbits ϑ, partitions Ω, with the equivalence relation:

$$\omega_1 = (\mu_1, \sigma_1) \sim (\mu_2, \sigma_2) = \omega_2 \Longleftrightarrow \frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} .$$

For fixed $\Theta \in \vartheta$, we take as our selection function Ψ that function which selects the point in Θ for which $\sigma = 1$.

Again the geometric interpretation is clear:



Similarly in the Z-space, fixing $z_o = (t,1)$, the orbit generated by it is $K_{z_o} = \{(\bar{x},s) : t = c\}$ which again represents a ray through the origin in the $(\bar{x},s)$ plane.

It is clear that t and δ represent maximal invariants in Z and Ω respectively. We shall make use of this fact below, as we shall identify orbits and maximal invariants.

We now use some of the results stated in Section 3 above, based on the invariance principle. Consider the game $G^* = (\mathcal{G} \times \Omega, \mathcal{N}, \rho^*)$ which was shown to be equivalent to G. By restricting ourselves to invariant procedures and making use of the simplification yielded by $g_A(a) = a$, we had obtained:

$$\rho^*(e, \Theta, \eta) = \int_{\mathcal{X}} \int_A L((\delta, 1), a)\, d\eta_K(a)\, dQ(K|\Theta) .$$

Now identifying the orbits K and Θ with the maximal invariants t and δ, we may consider the risk $\overline{\rho}^*$, say, where

$$\overline{\rho}^*(\delta,\overline{\eta}) = \int_{t=-\infty}^{+\infty}\int_A L((\delta,1),a)d\overline{\eta}(a|t)dP(t|\delta) \quad .$$

From this form of the risk function, it is clear that we have reduced the problem to the case where we are considering a statistical game in which the statistician observes a "t" -- e.g., a real-valued random variable, and nature chooses a "δ", again a real-valued parameter. Since $p(t'|\delta)$ has a monotone likelihood ratio (where $t = \frac{1}{\sqrt{N(N-1)}}t'$), we can conclude that if L satisfies the conditions set forth in the theorem, an essentially complete class of invariant procedures is the class of monotone procedures in terms of the t statistic. We consider some examples.

(1) Suppose we want to test the hypotheses:

$$H_o: \mu < 0 \qquad \text{vs.} \qquad H_1: \mu \geq 0 \quad .$$

Here $A = (a_o, a_1)$, e.g., accept H_o or H_1.

Our above result tells us that the classical procedure:

$$\text{Take action} \begin{Bmatrix} a_o & \text{if} & t < t_o \\ a_1 & \text{if} & t \geq t_o \end{Bmatrix} \quad ,$$

which is clearly a monotone procedure, does form an essentially complete class.

(2) We can consider an estimation problem, where we have N independent observations, $x_1,\ldots,x_N$, form a population with distribution $N(\mu,\sigma^2)$. It is required to obtain an estimate of the quantity

$$p = \frac{1}{\sqrt{2\pi}\,\sigma}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad .$$

Letting $y = \frac{x-\mu}{\sigma}$, we obtain

$$p = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\infty} e^{-y^2/2} dy \quad .$$

Hence p depends on ω only through δ. Here the action space A is the closed interval [0,1]. If we take the loss function of the form $L(\omega, a) = \frac{1}{f(p)} [\hat{p}-p]^2$ where $\hat{p}$ is the estimate and $f(p) > 0$ for all p, then our group $\mathcal{G}_1$ is again admissible and we can conclude that an essentially complete class of invariant procedures consists of estimates $\hat{p}$ which are monotone functions of t.

(3) We consider a slight generalization of Example (1).

Let $I_j = (\delta_j, \delta_{j+1}]$, $j = 1,\ldots,k$ where $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_{k+1}$ constitute a partition of the δ-axis into subintervals. We want to test H_j: $\delta \in I_j$, $j = 1,\ldots,k$. If $L((1,\delta),i)$ is of the form described earlier, then we again have the result that an essentially complete class of procedures is given by the procedures which are monotone.

Clearly (2) and (3) are examples of the problem considered under the Student Hypothesis, e.g., decisions concerning the proportion of the population falling beyond or below a certain value, or what is equivalent, decisions concerning μ/σ; example (1) is the classical problem, and is a special case of the others.

We shall next consider another group, and an application connected with it.

Consider the group $\mathcal{G}_2$, where

$$g_Z(\bar{x}, s) = g\bar{x} \ , \ |g|s$$

$$g_\Omega(\mu, \sigma) = g\mu \ , \ |g|\sigma$$

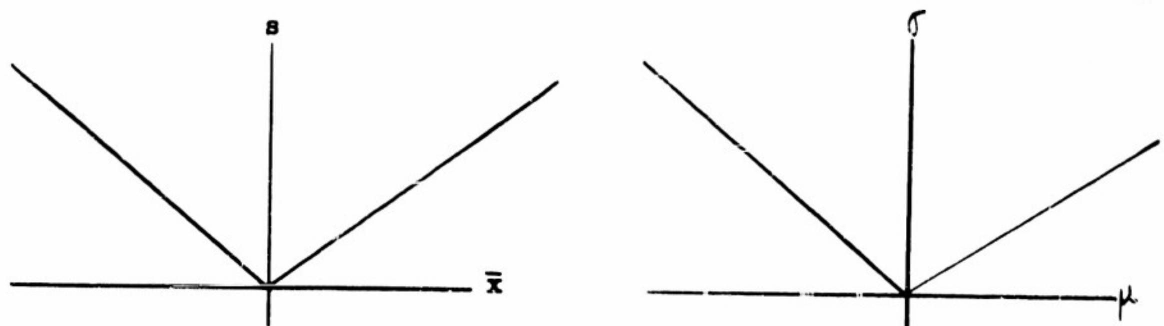
$$g_A(a) = a \qquad , \quad -\infty < g < \infty \quad , \quad g \neq 0 \ .$$

Assume that when operating with this group, the loss L is of the form $L(\omega,a) = L(|\frac{\mu}{\sigma}|,a)$.

For this group, a maximal invariant on Z and Ω respectively, are

$$\frac{|\bar{x}|}{\sqrt{\sum(x_i-x)^2}} \quad \text{and} \quad \left|\frac{\mu}{\sigma}\right|$$

The geometric representation of the orbits in Z and Ω are the 'reflected' rays $|t| = c$ and $|\delta| = c$, i.e.,



Since $\frac{\bar{x}}{\sqrt{\sum_{1}^{N} (x_i-\bar{x})^2}} > 0$, we may equivalently use the statistic

$t^2 = \frac{\bar{x}^2}{\sum(x_i-\bar{x})^2}$ which has (properly normalized) the non-central F distribution. Hence we may apply the monotonicity property of this distribution to construct essentially complete classes of invariant procedures with respect to this group.

So far we have considered examples for which we assumed $g_A(a) = a$; this we recall leads to considerable simplification in the expression for ρ^*, and makes it possible, in certain cases, to apply the complete class theorem directly. Let us now discuss an example in which the situation is more complex. Consider the group $\mathcal{J}_3$, defined by

$$g_Z(\bar{x},s) = g\bar{x}\ ,\ |g|\,s$$
$$g_\Omega(\mu,\sigma) = g\mu\ ,\ g\sigma$$
$$\begin{aligned} g_A(a) &= a \qquad \text{if } g > 0 \\ &= 1-a \quad \text{if } g < 0\ . \end{aligned}$$

This type of transformation is of interest when we consider the estimation problem.

We define two subgroups of $\mathcal{G}_3$:

$$\mathcal{H}_1:\ \left\{g \in \mathcal{G}_3:\ g > 0\right\}$$
$$\mathcal{H}_2:\ \left\{g \in \mathcal{G}_3:\ g = \pm 1\right\} \qquad ;$$

we note that $\mathcal{H}_1$ is the same as $\mathcal{G}_1$ considered before. We proceed as follows to construct an essentially complete class of decision procedures.

Let φ be any invariant decision procedure (with respect to $\mathcal{G}_3$); then φ is invariant with respect to $\mathcal{H}_1$, since $\mathcal{H}_1 \subseteq \mathcal{G}_3$. Since $\mathcal{H}_1 = \mathcal{G}_1$, we have from the previous example, that φ is a function of t.

It is proved in [7] that given any procedure φ, defined for a real-valued random, and invariant with respect to $\mathcal{H}_2$ (i.e., assume $\mathcal{H}_2$ to be operating on t and δ, changing $t \to \pm t$ and $\delta \to \pm \delta$, and $a \to \left\{\begin{matrix} a \\ 1-a \end{matrix}\right\}$, there exists a procedure φ', invariant and monotone, so that $\rho(\delta,\varphi') \le \rho(\delta,\varphi)$, all δ.

Hence, any φ invariant with respect to $\mathcal{G}_3$ is invariant with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$, and being a function of t, we may apply above result and obtain the result that for $\mathcal{G}_3$, the class of invariant, monotone (in t) procedures forms an essentially complete class.

We shall next consider the following application to the multi-variate case.

Let x be a p-dimensional random variable distributed according to the multi-variate normal law $N(\mu, \Sigma)$, where μ is the vector of expectations and Σ the covariance matrix (assumed to be non-singular). Let $x_1, \ldots, x_n$ be a sample of size n of x and let $\bar{x}$ be the vector of sample means and $S = X'X - n\bar{x}\bar{x}'$, where X is the matrix of observations. Since $\bar{x}$ and S are jointly sufficient for μ and Σ we need only consider the sample space $\mathfrak{Z} = (Z, \mathcal{B}, \Omega, P)$ where $Z = (\bar{x}, S)$. We wish to test the hypotheses:

$$H_0: \quad \mu = 0$$

$$H_1: \quad \mu \neq 0 \quad .$$

Assume the loss L to be of the form $L(\mu, \Sigma, a) = L^*(\mu' \Sigma^{-1} \mu, a)$. Consider the group $\mathcal{G}_4$: all $p \times p$ non-singular matrices, and define the following operation:

$$g_Z(\bar{x}, S) = g\bar{x} \ , \ gSg'$$

$$g_\Omega(\mu, \Sigma) = g\mu \ , \ g\Sigma g'$$

$$g_A(a) = a \quad .$$

We consider the statistic $F = \bar{x}' S^{-1} \bar{x}$ and shall show that it is a maximal invariant with respect to the group defined above.

F is clearly invariant; to show that it is maximal, we must show that any invariant function depends only on F. Assume that $\bar{x} \neq 0$. Let $z_0 = (\bar{x}_0, S_0)$ be a fixed element of Z, and let $K_{z_0} = \{z: z = g_Z(z_0), \text{ some } g\}$, be its orbit. Then K_{z_0} contains an element of the form $\begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \end{pmatrix}, I$ where $w \geq 0$ and I is the identity. This is so, since:

We can find a $g \in \mathcal{G}$, so that $gS_0 g' = I$. Also there is an orthogonal $h \in \mathcal{G}_3$ with $hg\bar{x}_0 = \begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; hence $hgS_0(hg)' = hIh' = I$ and thus the element

$k = hg \in \mathcal{G}_3$ maps z_o into $\begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \end{pmatrix}, I.$

Thus since invariant functions are constant over each orbit, we see that any invariant function depends only on w.

Evaluating w^2, with the above h and g, we have:

$$\begin{aligned} w^2 &= \bar{x}'g'h'[h(gSg')^{-1}h']hg\bar{x} \\ &= \bar{x}'g'h'hg'^{-1}S^{-1}g^{-1}h'hg\bar{x} \\ &= \bar{x}'S^{-1}\bar{x} \quad . \end{aligned}$$

Hence any invariant function depends only on w^2; i.e., $F = \bar{x}'S^{-1}\bar{x}$ is a maximal invariant.

Since F, properly normalized, has a non-central F distribution, we can again obtain the characterization of an essentially complete class, as a class of monotone procedures in F.

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